Sampling the Radon transform: theory and experiments

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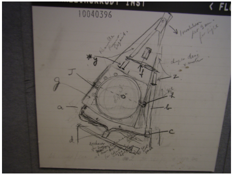
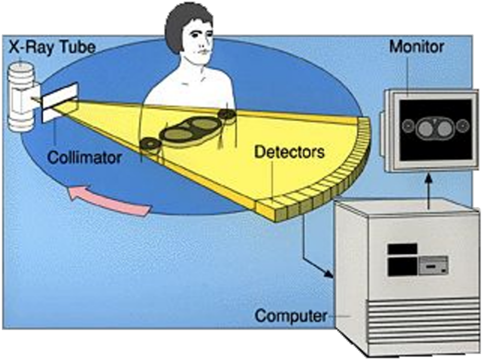
Outline

1. CT, Radon transform and its inversion
   - CT
   - Radon Transform
   - Radon and Fourier
2. Sampling
   - Sampling on lattices
   - Sampling the Radon Transform
3. Less Data
   - Local approaches
   - Compressive Sensing
   - ROI approaches

CT scanner: principle





$$I_D = I_S e^{-\int_S^D \mu(l)dl}$$

Cormack, Hounsfield, 79 Nobel Prize,
[Cormack(1963), Hounsfield(1973)].

From Radiology to Tomography

Definition

Let $\mu \in L^1\left(\mathbb{R}^2\right)$ then the Radon transform $\mathcal{R}$ of μ is defined by:

$$\mathcal{R}_\phi \mu\left(s\right) \stackrel{\text{def}}{=} \mathcal{R}\mu\left(\phi, s\right) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \mu\left(s\vec{\theta} + l\vec{\zeta}\right) dl \quad (1)$$

where $s \in \mathbb{R}$, $\vec{\theta}(\phi), \vec{\zeta}(\phi) \in \mathcal{S}^1$, the unit circle in $\mathbb{R}^2$,

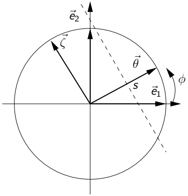


Figure: 2D tomography: parallel geometry parameters.

Center Slice Theorem and FBP

Theorem

Let $\mu \in \mathbb{L}^1(\mathbb{R}^2)$ *then*

$$\widehat{\mathcal{R}_\phi \mu}(\sigma) = \hat{\mu}(\sigma\vec{\theta})$$

Let $\mu \in \mathbb{L}^1(\mathbb{R}^2)$ *sufficiently smooth then*

$$\mu(\vec{x}) = \int_0^\pi \int_{\mathbb{R}} \widehat{\mathcal{R}_\phi \mu}(\sigma)|\sigma| e^{2i\pi\sigma\vec{x}\cdot\vec{\theta}} d\sigma d\phi$$

Petersen-Middleton - Shannon generalization

Let $g \in C_0^\infty(\mathbb{R}^n)$, the F.T. of g

$$\hat{g}(\vec{\xi}) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}} g(\vec{x}) e^{-i\vec{x}\cdot\vec{\xi}} d\vec{x}$$

Let $\mathbb{K} \subset \mathbb{R}^n$, the non-overlapping Shannon condition associated to $\mathbb{K}$ for the sampling lattice $L_W = W\mathbb{Z}^n$generated by the non singular $n \times n$ matrix W is that *the sets* $\mathbb{K} + 2\pi W^{-t} l, l \in \mathbb{Z}^n$ *are disjoint sets in* $\mathbb{R}^n$. The Petersen-Middleton theorem yields the Fourier interpolation formula

$$(S_W g)(\vec{x}) = \frac{1}{\sqrt{2\pi}^n} |\det W| \sum_{\vec{y} \in L_W} g(\vec{y}) \check{\chi}_{\mathbb{K}}(\vec{x} - \vec{y}),$$

where $\chi_{\mathbb{K}}$ is the indicator function of the set $\mathbb{K}$.
See [Petersen and Middleton(1962), Faridani(1994)].

Petersen-Middleton - Shannon generalization

The interpolation error is given by

$$||S_W g - g||_\infty \leq \frac{2}{\sqrt{2\pi}^n} \int_{\xi \notin \mathbb{K}} |\hat{g}(\xi)| d\xi.$$

$$W_I = h \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} ; 2\pi W_I^{-t} = \frac{\pi}{h} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} W_S = h \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

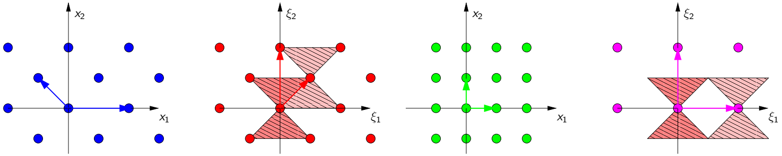


Figure: 2D interlaced and parallel sampling.

FT of the RT: essential support

Let $g \in \mathbb{L}^1([0, 2\pi[\times\mathbb{R})$,
$\hat{g}(k, \sigma) = (2\pi)^{-3/2} \int g(\phi, s) e^{-i(k\phi+\sigma s)} d\phi ds,\ k \in \mathbb{Z}, \sigma \in \mathbb{R}$.
Let $\mu \in \mathcal{C}_0^\infty(\Omega)$, $p(\phi, s) \stackrel{\mathrm{def}}{=} \mathcal{R}\mu(\phi, s)$ then

$$\sum_k \int_{(k,\sigma)\notin\mathbb{K}_\mathcal{R}} |\hat{p}(k, \sigma)| d\sigma < \eta(\vartheta, b) ||\mu||_{\mathbb{L}^1} + \frac{8}{\vartheta\sqrt{2\pi}} \epsilon_0(\mu, b)$$

where $\epsilon_0(\mu, b) = \int_{|\vec{\xi}|>b} \left|\hat{\mu}\left(\vec{\xi}\right)\right| d\vec{\xi}$ and $\eta(\vartheta, b) < e^{-c(\vartheta)b}$

$$\mathbb{K}_\mathcal{R} = \left\{(k, \sigma) \in \mathbb{Z} \times \mathbb{R}; |\sigma| < b; |k| < \max\left(\frac{|\sigma|}{\vartheta}, \left(\frac{1}{\vartheta} - 1\right) b\right)\right\}$$

[Rattey and Lindgren(1981), Natterer(1986)]

$\widehat{\mathcal{R}\mu}$ essential support

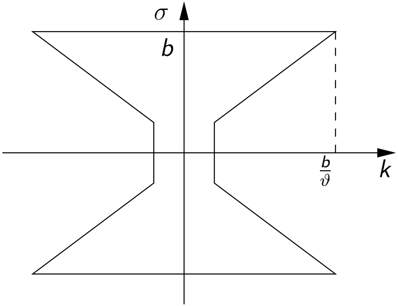


Figure: Essential support of the FT of the RT of a b-band limited function.

Standard Sampling

$$W_S = \begin{bmatrix} \frac{\pi}{P} & 0 \\ 0 & \frac{2}{Q} \end{bmatrix} \; ; \; 2\pi W_S^{-t} = \begin{bmatrix} 2P & 0 \\ 0 & \pi Q \end{bmatrix}$$

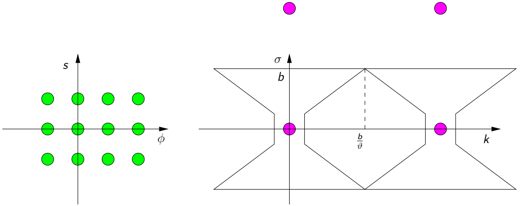


Figure: 2D parallel sampling.

Sampling conditions: $P > \frac{b}{\vartheta}$; $Q > 2\frac{b}{\pi}$.
Optimal relation: P slightly larger than $\frac{\pi}{2}Q$

(Efficient) Interlaced Sampling

$$W_I = \begin{bmatrix} \frac{2\pi}{P} & -\frac{\frac{\pi}{P}}{} \\ 0 & \frac{2}{Q} \end{bmatrix} \; ; \; 2\pi W_I^{-t} = \begin{bmatrix} P & 0 \\ \frac{\pi}{2}Q & \pi Q \end{bmatrix}$$

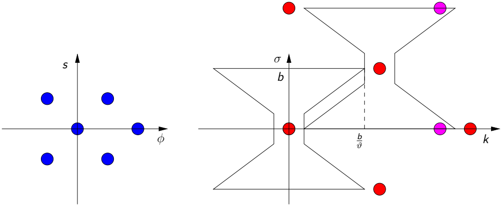


Figure: 2D parallel sampling.

Sampling conditions: $Q > 2\frac{b}{\pi}$; $P = \pi\frac{Q}{\vartheta'}$
Optimal relation: P slightly larger than πQ

Generalization in 2D: Fan Beam Geometry

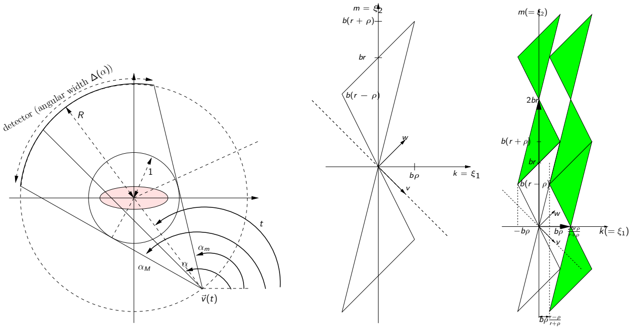


Figure: 2D Fan Beam

[Natterer(1996), Natterer(1999), Gratton.(2005), Izen et al.(2005)Izen, Rohler, and Sastry, Faridani(2006)]

Generalization in 3D: Parallel Beam Geometry

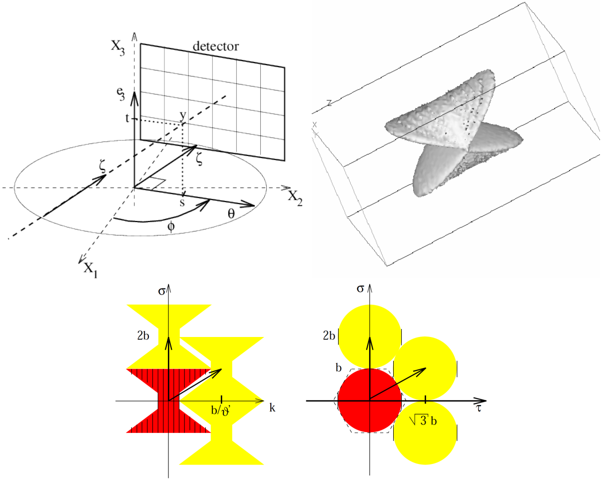


Figure: 3D Parallel Beam [Desbat(1997)]

Generalization in 3D: Fan Beam Geometry

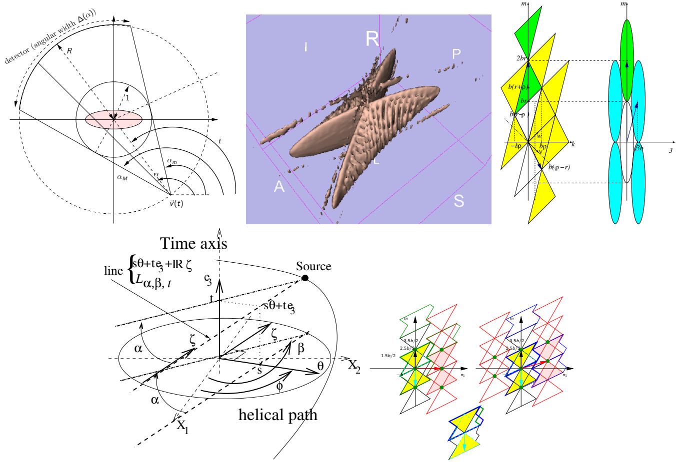


Figure: 3D Fan Beam X-ray [Desbat and Gratton(2007)], [Desbat et al.(2004)Desbat, Roux, Koennig, and Grangeat]

Generalizations and applications

- SPECT collimation geometries (nuclear imaging)
- Vector field tomography [Desbat(1995)]
- Doppler Imaging in astrophysics (generalized rotation invariant RT) [Desbat and Mennessier(1997)]
- Efficient scheme perturbations for industrial applications [Desbat(1993)]
- ...

Non Local Filter

We define p_H the Hilbert transform of the parallel projection p

$$p_H(\phi, s) = \int_{-\infty}^{+\infty} p(\phi, t) h(s - t) dt \text{ where } h(u) = \frac{1}{\pi u}$$

and $\hat{h}(\sigma) = -i\mathrm{sgn}(\sigma)$ (distribution). The ramp filtering p_R of p is

$$p_R(\phi, s) = \frac{1}{2\pi} \frac{\partial}{\partial s} p_H(\phi, s)$$

The filter $p_\phi(s) \xrightarrow{\mathcal{F}} \widehat{p_\phi}(\sigma) \xrightarrow{\text{filter}} \widehat{p_\phi}(\sigma)|\sigma| \xrightarrow{\mathcal{F}^{-1}} p_{\phi_{\text{filtered}}}(s)$ is non local !!! $|\sigma| = \frac{1}{2\pi}(2i\pi\sigma)(-i\mathrm{sgn}(\sigma))$ is the Hilbert filtering composed by the derivation.

Λ-tomography ; wavelets

- Reconstruct $\Lambda\mu$ locally from p where

$$\widehat{\Lambda\mu}\left(\vec{\xi}\right) = \hat{\mu}\left(\vec{\xi}\right)\left|\vec{\xi}\right|$$

  [Faridani et al.(1992)Faridani, Ritman, and Smith, Bilgot et al.(2011)Bilgot, Desbat, and Perrier]
- Wavelets
  [Delaney and Bresler(1995), Berenstein and Walnut(1996), Bonnet et al.(2000)Bonnet, Peyrin, Turjman, and Prost, Bilgot(2007)]

Compressive Sensing

- Assumption μ is compressive (e.g., can be represented by few coefficients in a wavelet base) then μ can be identified (with high probability) from few projection samples

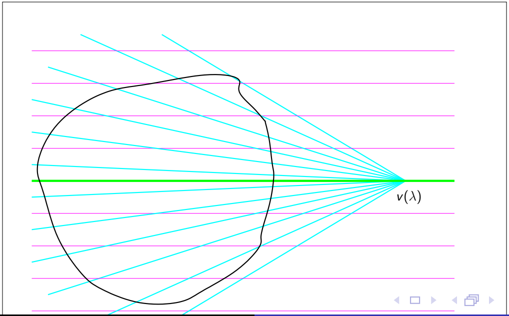
[Candes and Wakin(2008), Wang(2011)]

The parallel Fan Beam Hilbert Projection Equality

Theorem

$$p_H(\phi, v_t \cdot \vec{\theta}) = g_H(v_t, \phi) \qquad (2)$$

(idea compute $p_H(\phi, s)$ from $g_H(v_t, \phi)$ with $v_t \cdot \vec{\theta} = s$) [Noo et al.(2002)Noo, Defrise, Clackdoyle, and Kudo, Clackdoyle and Defrise(2010)]



Very Short Scan

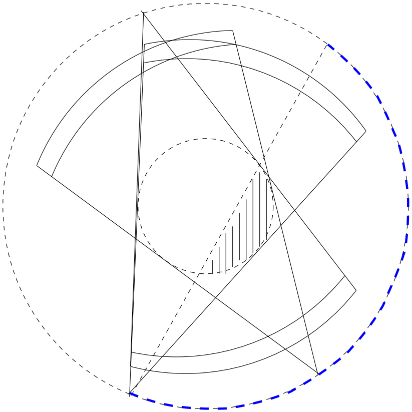


Figure: Very Short Scan....

Virtual Fean Beam

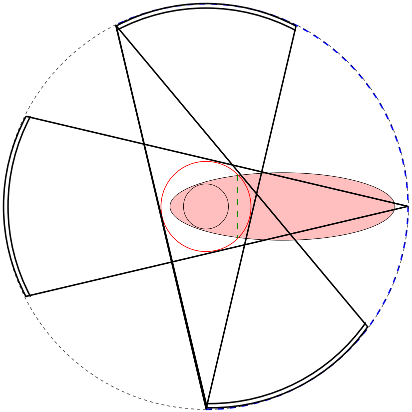
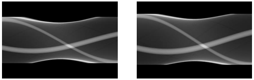
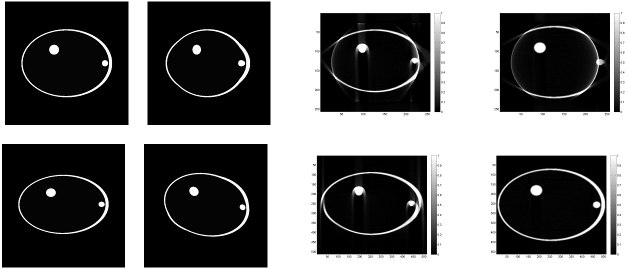


Figure: Data truncation: Hip....

Dynamic and ROI Tomography

$$\mu_{\vec{\Gamma}_t}(\vec{x}) \stackrel{\text{def}}{=} \mu\left(\vec{\Gamma}_t(\vec{x})\right)\left|\det J_{\vec{\Gamma}_t}(\vec{x})\right|$$





ROI

FBP

Left: dyn. phantom ; right: 1rst Col. NoCorr, 2nd Col. Dyn.Corr [Desbat et al.(2007)Desbat, Roux, and Grangeat, Desbat et al.(2012)Desbat, Mennessier, and Clackdoyle]

Thank you! Questions ?

Thank you for your attention!
Questions ?

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